PROBLEM 1

1. Derive the 2D Fourier transform of a 2D *rect* function, which is defined as below. Show your work.

**ANSWER**

$$\mathrm{rect}\left(\frac{x}{X},\frac{y}{Y}\right)=\begin{cases}1, & |x|\le\frac{X}{2},|y|\le\frac{Y}{2}\\ 0, & \text{otherwise}\end{cases} \tag{1}$$

1D Fourier transform is defined as

$$F(\omega)=\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \tag{2}$$

and in 2D case it can be defined as

$$F(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-i(ux+vy)}dxdy \tag{3}$$

We can write given function from Eq. (1) as

$$f(x,y)=g(x)h(y) \tag{4}$$

It follows using Eq. (3)

$$\begin{aligned}F(u,v)&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-i(ux+vy)}dxdy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)e^{-iux}e^{-ivy}dxdy\\&=\int_{-\infty}^{\infty} g(x)e^{-iux}dx\int_{-\infty}^{\infty} h(y)e^{-ivy}dy\\&=\left[\int_{-\infty}^{-X/2} g(x)e^{-iux}dx+\int_{-X/2}^{X/2} g(x)e^{-iux}dx+\int_{X/2}^{\infty} g(x)e^{-iux}dx\right]\\&\times\left[\int_{-\infty}^{-Y/2} h(y)e^{-ivy}dy+\int_{-Y/2}^{Y/2} h(y)e^{-ivy}dy+\int_{Y/2}^{\infty} h(y)e^{-ivy}dy\right]\end{aligned} \tag{5}$$

In region $-\infty$ to $-X/2$ and $X/2$ to $\infty$, function $g(x)=0$ and first and third integral will be zero. Similarly for $h(y)$. We have

$$
\begin{aligned}
F(u,v) &= \int_{-X/2}^{X/2} g(x)e^{-iux}dx \int_{-Y/2}^{Y/2} h(y)e^{-ivy}dy = \int_{-X/2}^{X/2} e^{-iux}dx \int_{-Y/2}^{Y/2} e^{-ivy}dy \\
&= \left( \frac{1}{-iu} e^{-iux} \Big|_{-X/2}^{X/2} \right)\left( \frac{1}{-iv} e^{-ivy} \Big|_{-Y/2}^{Y/2} \right) \\
&= \frac{1}{-iu}\left( e^{-iu\frac{X}{2}} - e^{iu\frac{X}{2}} \right)\frac{1}{-iv}\left( e^{-iv\frac{Y}{2}} - e^{iv\frac{Y}{2}} \right) \\
&= \frac{1}{-iu}\left[ \cos\left( u\frac{X}{2} \right) - i\sin\left( u\frac{X}{2} \right) - \cos\left( u\frac{X}{2} \right) - i\sin\left( u\frac{X}{2} \right) \right]\frac{1}{-iv}\left( e^{-iv\frac{Y}{2}} - e^{iv\frac{Y}{2}} \right) \\
&= \frac{2}{u}\sin\left( u\frac{X}{2} \right)\frac{2}{v}\sin\left( v\frac{Y}{2} \right)
\end{aligned} \tag{6}
$$

We have obtained 2D Fourier transform of rect function given in Eq. (1). It can be also written in terms of sinc function which is defined as

$$
\operatorname{sinc}(ax) = \frac{\sin(ax)}{ax} \tag{7}
$$

We have

$$
\begin{aligned}
F(u,v) &= \frac{2}{u}\sin\left( u\frac{X}{2} \right)\frac{2}{v}\sin\left( v\frac{Y}{2} \right) = XY\frac{\sin\left( u\frac{X}{2} \right)}{u\frac{X}{2}}\frac{\sin\left( v\frac{Y}{2} \right)}{v\frac{Y}{2}} \\
&= XY\operatorname{sinc}\left( u\frac{X}{2} \right)\operatorname{sinc}\left( v\frac{Y}{2} \right)
\end{aligned} \tag{8}
$$

PROBLEM 2

2. Consider a well-behaved continuous, integrable time signal $f(t)$ that is band-limited, *i.e.* $F(\omega) = \mathcal{F}\{f(t)\} = 0$ for $|\omega| > B/2$.
Consider that $f_n = f(n\Delta t)$ is an appropriately sampled version of $f(t)$ at intervals $\Delta t$ such that the Nyquist sampling criterion is respected, *i.e.* $\Delta t \geq 1/B$.

a. Derive an expression for, and describe, the (continuous) Fourier transform of the sampled function $f_n$, *i.e.* the effect of sampling on the Fourier information.
b. *Zero-padding* is the action of "extending" the information in the Fourier domain, beyond a certain frequency, by adding zeros. (Alternatively, you might view this as *replacing* the information with zeros…) Derive an expression for the function $g_n = f\left(n\frac{\Delta t}{2}\right)$ that is obtained by zero-padding the Fourier transform of $f_n$ to an appropriately large frequency $|\omega| \to \infty$, and taking the inverse Fourier transform of that result.
c. Discuss the time-domain interpretation of the zero-padding operation, and the implications with respect to recovering the signal $f(t)$ from $f_n$.

**ANSWER**

**a.**

Sampling the signal $f(t)$ involves multiplying that signal with the impulse train (also knows as comb or shah) function which is defined as

$$Ш_{\Delta t}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta t) \tag{9}$$

Sampled signal will be thus

$$\begin{aligned} f_n = f(n\Delta t) &= f(t)\,Ш_{\Delta t}(t) \\ &= \sum_{n=-\infty}^{\infty} f(t)\delta(t-n\Delta t) \\ &= \sum_{n=-\infty}^{\infty} f(n\Delta t)\delta(t-n\Delta t) \end{aligned} \tag{10}$$

Because comb function is periodic, we can find its Fourier series representation

$$Ш_{\Delta t}(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\frac{2\pi}{\Delta t}t} = \frac{1}{\Delta t}\sum_{n=-\infty}^{\infty} e^{in\frac{2\pi}{\Delta t}t} \tag{11}$$

Coefficients $c_n$ are found as

$$c_n = \frac{1}{\Delta t}\int_{-\Delta t/2}^{\Delta t/2} Ш_{\Delta t}(t) e^{-in\frac{2\pi}{\Delta t}t} dt = \frac{1}{\Delta t}\int_{-\Delta t/2}^{\Delta t/2} \sum_{n=-\infty}^{\infty} \delta(t - n\Delta t) e^{-in\frac{2\pi}{\Delta t}t} dt$$
$$= \frac{1}{\Delta t}\sum_{n=-\infty}^{\infty} \int_{-\Delta t/2}^{\Delta t/2} \delta(t - n\Delta t) e^{-in\frac{2\pi}{\Delta t}t} dt = \int_{-\Delta t/2}^{\Delta t/2} \delta(t) e^{-in\frac{2\pi}{\Delta t}t} dt = \frac{1}{\Delta t} \qquad (12)$$

where we used the fact that function $\delta(t - n\Delta t)$ is zero in interval $\left[-\frac{\Delta t}{2}, \frac{\Delta t}{2}\right]$ in case when $n \neq 0$.

Knowing that $F\left\{e^{in\frac{2\pi}{\Delta t}t}\right\} = \delta\left(\omega - \frac{n}{\Delta t}\right)$, it follows that Fourier transform of comb function is

$$F\left\{Ш_{\Delta t}(t)\right\} = \frac{1}{\Delta t}\sum_{n=-\infty}^{\infty} F\left\{e^{in\frac{2\pi}{\Delta t}t}\right\} = \frac{1}{\Delta t}\sum_{n=-\infty}^{\infty} \delta\left(\omega - \frac{n}{\Delta t}\right) \qquad (13)$$

Multiplication in time domain is equal to convolution in frequency domain. That means when taking Fourier transform of sampled function we will have convolution of them. It follows

$$f_n = F\left\{f(t) Ш_{\Delta t}(t)\right\} = F\left\{f(t)\right\} * F\left\{Ш_{\Delta t}(t)\right\}$$
$$= \sum_{n=-\infty}^{\infty} F\left\{f(n\Delta t)\right\} * F\left\{\delta(t - n\Delta t)\right\}$$
$$= F(\omega) * \frac{1}{\Delta t}\sum_{n=-\infty}^{\infty} \delta\left(\omega - \frac{n}{\Delta t}\right) = \frac{1}{\Delta t}\sum_{n=-\infty}^{\infty} F(\omega) * \delta\left(\omega - \frac{n}{\Delta t}\right) \qquad (14)$$
$$= \frac{1}{\Delta t}\sum_{n=-\infty}^{\infty} F\left(\omega - \frac{n}{\Delta t}\right)$$

where shifting property of delta function was used when finding convolution of two functions.

The final expression Eq. (14) shows that Fourier transform of the sampled function is a periodic function consisting of the repeated copies of the transform of the original continuous-time signal.

**b.**

We have that discrete Fourier transform (DFT) is

$$f_n = \frac{1}{\tau}\sum_{t=0}^{\tau-1} f_t e^{-in\frac{2\pi}{\tau}t} \qquad (15)$$

and inverse DFT

$$F_n = \sum_{n=0}^{N-1} f_n e^{in\frac{2\pi}{\tau}t} \quad (16)$$

In our case we have that $\tau = \frac{\Delta t}{2}$. Now, if we add M zero points at the end of our signal, we get

$$f_n = \frac{1}{\tau}\sum_{t=0}^{\tau+M-1} f_t e^{-in\frac{2\pi}{\tau+M}t} = \frac{1}{\tau}\sum_{t=0}^{\tau-1} f_t e^{-in\frac{2\pi}{\tau+M}t} \quad (17)$$

The sum doesn't change because extra zeroes don't contribute to it. Now, we have $\tau + M$ spectral samples with the same Nyquist frequency but with different line spacing. Taking the inverse DFT, we get

$$\begin{aligned} g_n = F_n &= \sum_{n=0}^{N-1} f_n e^{in\frac{2\pi}{\tau}t} = \sum_{n=0}^{N-1}\left(\frac{1}{\tau}\sum_{t=0}^{\tau-1} f_t e^{-in\frac{2\pi}{\tau+M}t}\right) e^{in\frac{2\pi}{\tau}t} \\ &= \sum_{t=0}^{\tau-1} f_t \frac{1}{\tau}\sum_{n=0}^{N-1} e^{in\left(\frac{2\pi t}{\tau}-\frac{2\pi t}{\tau+M}\right)} \end{aligned} \quad (18)$$

The term $\frac{1}{\tau}\sum_{n=0}^{N-1} e^{in\left(\frac{2\pi t}{\tau}-\frac{2\pi t}{\tau+M}\right)}$ can be solved analytically which leads to sinc function (aliased sinc function, https://en.wikipedia.org/wiki/Dirichlet_kernel).

**c.**

Zero padding means added extra zeros onto the end of $f_t$ before performing the DFT. Because the zero-padded signal is longer the resulting DFT provides better frequency resolution. Zero-padding in the time-domain results in interpolation in the frequency-domain.

PROBLEM 3

3. Imagine the following system, which takes one input signal and returns two output signals:

The system is tested with various inputs and returns the following outputs:

| Input | Output |
|---|---|
| $f(t) = \cos(\omega t)$ | $h_1(t) = A\cos(\omega t), h_2(t) = B\sin(\omega t)$ |
| $f(t) = \sin(\omega t)$ | $h_1(t) = A\sin(\omega t), h_2(t) = -B\cos(\omega t)$ |

where $A$ and $B$ are real constants, and $\omega$ is any real frequency.

Given the information above, derive a possible system transfer function $G(\omega)$, and the related impulse response function $g(t)$. Show that this is a linear time-invariant system.

Note: $\cos(\omega t - \pi/2) = \sin(\omega t)$ and $\sin(\omega t - \pi/2) = -\cos(\omega t)$

**ANSWER**

possible system transfer function:

Output signals:

$$y_1 = Ax$$

$$y_2 = B\cos\left(\omega t - \frac{\pi}{2}\right) = Bx\left(\omega t - \frac{\pi}{2}\right)$$

Fourier transform of output signals

$$Y_1(\omega) = AX(\omega)$$

$$Y_2(\omega) = Be^{-i\omega\frac{\pi}{2}}X(\omega)$$

Transfer function

$$G(\omega)=\begin{bmatrix} G_1(\omega) \\ G_2(\omega) \end{bmatrix}$$

$$G_1(\omega)=\frac{Y_1(\omega)}{X(\omega)}=A$$

$$G_2(\omega)=\frac{Y_2(\omega)}{X(\omega)}=\frac{B}{j\omega}$$

$$G(\omega)=\begin{bmatrix} G_1(\omega) \\ G_2(\omega) \end{bmatrix}=\begin{bmatrix} A \\ Be^{-i\omega\frac{\pi}{2}} \end{bmatrix}$$

Impulse response function as inverse Fourier transform of transfer function

$$g(t)=[G(\omega)]^{-1}=\begin{bmatrix} A\delta(t) \\ B\delta\left(t-\frac{\pi}{2}\right) \end{bmatrix}$$

This is a a linear time-invariant system due to for unit sample signal we have unit sample response.

Let see that this is a linear time-invariant system for various inputs and corresponds outputs:

For the first input and output:

Input

$x(t)=\cos(\omega t)=-\sin(\omega t-\pi/2)$ - input is a sinusoidal signal

Output

$y_1(t)=A\cos(\omega t)=-A\sin(\omega t-\pi/2)$ - is a sinusoidal signal

$y_2(t)=B\sin(\omega t)$ - is a sinusoidal signal

For the second pair

Input

$x(t) = \sin(\omega t)$ - input is a sinusoidal signal

Output

$y_1(t) = A\sin(\omega t)$ - is a sinusoidal signal

$y_2(t) = -B\cos(\omega t) = B\sin(\omega t - \pi / 2)$ - is a sinusoidal signal